CHAPTER 1
NUMBER SYSTEMS AND SETS

Mathematics is a basic tool. Some use of mathematics is found in every rating in the Navy, from the simple arithmetic of counting for inventory purposes to the complicated equations encountered in computer and engineering work. Storekeepers need mathematical computation in their bookkeeping. Damage Controlmen need mathematics to compute stress, centers of gravity, and maximum permissible roll. Electronics principles are frequently stated by means of mathematical formulas. Navigation and engineering also use mathematics to a great extent. As maritime warfare becomes more and more complex, mathematics achieves ever increasing importance as an essential tool.

From the point of view of the individual there are many incentives for learning the subject. Mathematics better equips him to do his present job. It will help him in attaining promotions and the corresponding pay increases. Statistically it has been found that one of the best indicators of a man's potential success as a naval officer is his understanding of mathematics.

This training course begins with the basic facts of arithmetic and continues through some of the early stages of algebra. An attempt is made throughout to give an understanding of why the rules of mathematics are true. This is done because it is felt that rules are easier to learn and remember if the ideas that led to their development are understood.

Many of us have areas in our mathematics background that are hazy, barely understood, or troublesome. Thus, while it may at first seem beneath your dignity to read chapters on fundamental arithmetic, these basic concepts may be just the spots where your difficulties lie. These chapters attempt to treat the subject on an adult level that will be interesting and informative.

COUNTING

Counting is such a basic and natural process that we rarely stop to think about it. The process is based on the idea of ONE-TO-ONE CORRESPONDENCE, which is easily demonstrated by using the fingers. When children count on their fingers, they are placing each finger in one-to-one correspondence with one of the objects being counted. Having outgrown finger counting, we use numerals.

NUMERALS

Numerals are number symbols. One of the simplest numeral systems is the Roman numeral system, in which tally marks are used to represent the objects being counted. Roman numerals appear to be a refinement of the tally method still in use today. By this method, one makes short vertical marks until a total of four is reached; when the fifth tally is counted, a diagonal mark is drawn through the first four marks. Grouping by fives in this way is reminiscent of the Roman numeral system, in which the multiples of five are represented by special symbols.

A number may have many "names." For example, the number 6 may be indicated by any of the following symbols: $9 - 3$, $12/2$, $5 + 1$, or 2×3. The important thing to remember is that a number is an idea; various symbols used to indicate a number are merely different ways of expressing the same idea.

POSITIVE WHOLE NUMBERS

The numbers which are used for counting in our number system are sometimes called natural numbers. They are the positive whole numbers, or to use the more precise mathematical term, positive INTEGERS. The Arabic numerals from 0 through 9 are called digits, and an integer may have any number of digits. For example, 5, 32, and 7,049 are all integers. The number of digits in an integer indicates its rank; that is, whether it is "in the hundreds," "in the thousands," etc. The idea of ranking numbers in terms of tens, hundreds, thousands, etc., is based on the PLACE VALUE concept.

PLACE VALUE

Although a system such as the Roman numeral system is adequate for recording the results of counting, it is too cumbersome for purposes of calculation. Before arithmetic could develop as we know it today, the following two important concepts were needed as additions to the counting process:

1. The idea of 0 as a number.
2. Positional notation (place value).

Positional notation is a form of coding in which the value of each digit of a number depends upon its position in relation to the other digits of the number. The convention used in our number system is that each digit has a higher place value than those digits to the right of it.

The place value which corresponds to a given position in a number is determined by the BASE of the number system. The base which is most commonly used is ten, and the system with ten as a base is called the decimal system (decem is the Latin word for ten). Any number is assumed to be a base-ten number, unless some other base is indicated. One exception to this rule occurs when the subject of an entire discussion is some base other than ten. For example, in the discussion of binary (base two) numbers later in this chapter, all numbers are assumed to be binary numbers unless some other base is indicated.

DECIMAL SYSTEM

In the decimal system, each digit position in a number has ten times the value of the position adjacent to it on the right. For example, in the number 11, the 1 on the left is said to be in the "tens place," and its value is 10 times as great as that of the 1 on the right. The 1 on the right is said to be in the "units place," with the understanding that the term "unit" in our system refers to the numeral 1. Thus the number 11 is actually a coded symbol which means "one ten plus one unit." Since ten plus one is eleven, the symbol 11 represents the number eleven.

Figure 1-1 shows the names of several digit positions in the decimal system. If we apply this nomenclature to the digits of the integer 235, then this number symbol means "two hundreds plus three tens plus five units." This number may be expressed in mathematical symbols as follows:

$$2 \times 10 \times 10 + 3 \times 10 + 5 \times 1$$

Notice that this bears out our earlier statement: each digit position has 10 times the value of the position adjacent to it on the right.

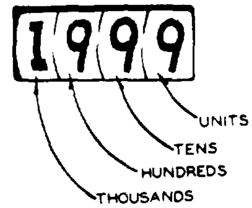

Figure 1-1.—Names of digit positions.

The integer 4,372 is a number symbol whose meaning is "four thousands plus three hundreds plus seven tens plus two units." Expressed in mathematical symbols, this number is as follows:

$$4 \times 1000 + 3 \times 100 + 7 \times 10 + 2 \times 1$$

This presentation may be broken down further, in order to show that each digit position as 10 times the place value of the position on its right, as follows:

$$4 \times 10 \times 100 + 3 \times 10 \times 10 + 7 \times 10 \times 1 + 2 \times 1$$

The comma which appears in a number symbol such as 4,372 is used for "pointing off" the digits into groups of three beginning at the right-hand side. The first group of three digits on the right is the units group; the second group is the thousands group; the third group is the millions group; etc. Some of these groups are shown in table 1-1.

Table 1-1.—Place values and grouping.

Billions group	Millions group	Thousands group	Units group
Hundred billions	Hundred millions	Hundred thousands	Hundreds
Ten billions	Ten millions	Ten thousands	Tens
Billions	Millions	Thousands	Units

By reference to table 1-1, we can verify that 5,432,786 is read as follows: five million, four hundred thirty-two thousand, seven hundred eighty-six. Notice that the word "and" is not necessary when reading numbers of this kind.

Practice problems:
1. Write the number symbol for seven thousand two hundred eighty-one.
2. Write the meaning, in words, of the symbol 23,469.
3. If a number is in the millions, it must have at least how many digits?
4. If a number has 10 digits, to what number group (thousands, millions, etc.) does it belong?

Answers:
1. 7,281
2. Twenty-three thousand, four hundred sixty-nine.
3. 7
4. Billions

BINARY SYSTEM

The binary number system is constructed in the same manner as the decimal system. However, since the base in this system is two, only two digit symbols are needed for writing numbers. These two digits are 1 and 0. In order to understand why only two digit symbols are needed in the binary system, we may make some observations about the decimal system and then generalize from these.

One of the most striking observations about number systems which utilize the concept of place value is that there is no single-digit symbol for the base. For example, in the decimal system the symbol for ten, the base, is 10. This symbol is compounded from two digit symbols, and its meaning may be interpreted as "one base plus no units." Notice the implication of this where other bases are concerned: Every system uses the same symbol for the base, namely 10. Furthermore, the symbol 10 is not called "ten" except in the decimal system.

Suppose that a number system were constructed with five as a base. Then the only digit symbols needed would be 0, 1, 2, 3, and 4. No single-digit symbol for five is needed, since the symbol 10 in a base-five system with place value means "one five plus no units." In general, in a number system using base N, the largest number for which a single-digit symbol is needed is N minus 1. Therefore, when the base is two the only digit symbols needed are 1 and 0.

An example of a binary number is the symbol 101. We can discover the meaning of this symbol by relating it to the decimal system. Figure 1-2 shows that the place value of each digit position in the binary system is two times the place value of the position adjacent to it on the right. Compare this with figure 1-1, in which the base is ten rather than two.

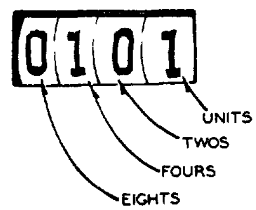

Figure 1-2.—Digit positions in the binary system.

Placing the digits of the number 101 in their respective blocks on figure 1-2, we find that 101 means "one four plus no twos plus one unit." Thus 101 is the binary equivalent of decimal 5. If we wish to convert a decimal number, such as 7, to its binary equivalent, we must break it into parts which are multiples of 2. Since 7 is equal to 4 plus 2 plus 1, we say that it "contains" one 4, one 2, and one unit. Therefore the binary symbol for decimal 7 is 111.

The most common use of the binary number system is in electronic digital computers. All data fed to a typical electronic digital computer is converted to binary form and the computer performs its calculations using binary arithmetic rather than decimal arithmetic. One of the reasons for this is the fact that electrical and electronic equipment utilizes many switching circuits in which there are only two operating conditions. Either the circuit is "on" or it is "off," and a two-digit number system is ideally suited for symbolizing such a situation.

Details concerning binary arithmetic are beyond the scope of this volume, but are available in Mathematics, Volume 3, NavPers 10073, and in Basic Electonics, NavPers 10087-A.

Practice problems:
1. Write the decimal equivalents of the binary numbers 1101, 1010, 1001, and 1111.
2. Write the binary equivalents of the decimal numbers 12, 7, 14, and 3.

Answers:
1. 13, 10, 9, and 15
2. 1100, 111, 1110, and 11

SETS

Any serious study of mathematics leads the student to investigate more than one text and more than one way of approaching each new topic. At the time of printing of this course, much emphasis is being placed on so-called modern math in the public schools. Consequently, the trainee who uses this course is likely to find considerable material, in his parallel reading, which uses the ideas and terminology of the "new" math.

In the following paragraphs, a very brief introduction to some of the set theory of modern math is presented. Although the remainder of this course is not based on set theory, this brief introduction should help in making the transition from traditional methods to newer, experimental methods.

DEFINITIONS AND SYMBOLS

The word "set" implies a collection or grouping of similar objects or symbols. The objects in a set have at least one characteristic in common, such as similarity of appearance or purpose. A set of tools would be an example of a group of objects not necessarily similar in appearance but similar in purpose. The objects or symbols in a set are called members or ELEMENTS of the set.

The elements of a mathematical set are usually symbols, such as numerals, lines, or points. For example, the integers greater than zero and less than 5 form a set, as follows:

$$\{1, 2, 3, 4\}$$

Notice that braces are used to indicate sets. This is often done where the elements of the set are not too numerous.

Since the elements of the set $\{2, 4, 6\}$ are the same as the elements of $\{4, 2, 6\}$, these two sets are said to be equal. In other words, equality between sets has nothing to do with the order in which the elements are arranged. Furthermore, repeated elements are not necessary. That is, the elements of $\{2, 2, 3, 4\}$ are simply 2, 3, and 4. Therefore the sets $\{2, 3, 4\}$ and $\{2, 2, 3, 4\}$ are equal.

Practice problems:

1. Use the correct symbols to designate the set of odd positive integers greater than 0 and less than 10.
2. Use the correct symbols to designate the set of names of days of the week which do not contain the letter "s".
3. List the elements of the set of natural numbers greater than 15 and less than 20.
4. Suppose that we have sets as follows:

$$A = \{1, 2, 3\} \qquad C = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 2, 3\} \qquad D = \{1, 1, 2, 3\}$$

Which of these sets are equal?

Answers:

1. $\{1, 3, 5, 7, 9\}$
2. {Monday, Friday}
3. 16, 17, 18, and 19
4. $A = B = D$

SUBSETS

Since it is inconvenient to enumerate all of the elements of a set each time the set is mentioned, sets are often designated by a letter. For example, we could let S represent the set of all integers greater than 0 and less than 10. In symbols, this relationship could be stated as follows:

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Now suppose that we have another set, T, which comprises all positive even integers less than 10. This set is then defined as follows:

$$T = \{2, 4, 6, 8\}$$

Notice that every element of T is also an element of S. This establishes the SUBSET relationship; T is said to be a subset of S.

POSITIVE INTEGERS

The most fundamental set of numbers is the set of positive integers. This set comprises the counting numbers (natural numbers) and includes, as subsets, all of the sets of numbers which we have discussed. The set of natural numbers has an outstanding characteristic: it is infinite. This means that the successive elements of the set continue to increase in size without limit, each number being larger by 1 than the number preceding it. Therefore there is no "largest" number; any number that we might choose as larger than all others could be increased to a larger number simply by adding 1 to it.

One way to represent the set of natural numbers symbolically would be as follows:

$$\{1, 2, 3, 4, 5, 6, \ldots\}$$

The three dots, called ellipsis, indicate that the pattern established by the numbers shown continues without limit. In other words, the next number in the set is understood to be 7, the next after that is 8, etc.

POINTS AND LINES

In addition to the many sets which can be formed with number symbols, we frequently find it necessary in mathematics to work with sets composed of points or lines.

A point is an idea, rather than a tangible object, just as a number is. The mark which is made on a piece of paper is merely a symbol representing the point. In strict mathematical terms, a point has no dimensions (physical size) at all. Thus a pencil dot is only a rough picture of a point, useful for indicating the location of the point but certainly not to be confused with the idea.

Now suppose that a large number of points are placed side by side to form a "string." Picturing this arrangement by drawing dots on paper, we would have a "dotted line." If more dots were placed between the dots already in the string, with the number of dots increasing until we could not see between them, we would have a rough picture of a line. Once again, it is important to emphasize that the picture is only a symbol which represents an ideal line. The ideal line would have length but no width or thickness.

The foregoing discussion leads to the conclusion that a line is actually a set of points. The number of elements in the set is infinite, since the line extends in both directions without limit.

The idea of arranging points together to form a line may be extended to the formation of planes (flat surfaces). A mathematical plane is determined by three points which do not lie on the same line. It is also determined by two intersecting lines.

Line Segments and Rays

When we draw a "line," label its end points A and B, and call it "line AB," we really mean LINE SEGMENT AB. A line segment is a subset of the set of points comprising a line.

When a line is considered to have a starting point but no stopping point (that is, it extends without limit in one direction), it is called a RAY. A ray is not a line segment, because it does not terminate at both ends; it may be appropriate to refer to a ray as a "half-line."

THE NUMBER LINE

As in the case of a line segment, a ray is a subset of the set of points comprising a line. All three—lines, line segments, and rays—are subsets of the set of points comprising a plane.

Among the many devices used for representing a set of numbers, one of the most useful is the number line. To illustrate the construction of a number line, let us place the elements of the set of natural numbers in one-to-one correspondence with points on a line. Since the natural numbers are equally spaced, we select points such that the distances between them are equal. The starting point is labeled 0, the next point is labeled 1, the next 2, etc., using the natural numbers in normal counting order. (See fig. 1-3.) Such an arrangement is often referred to as a scale, a familiar example being the scale on a thermometer.

Thus far in our discussion, we have not mentioned any numbers other than integers. The number line is an ideal device for picturing the

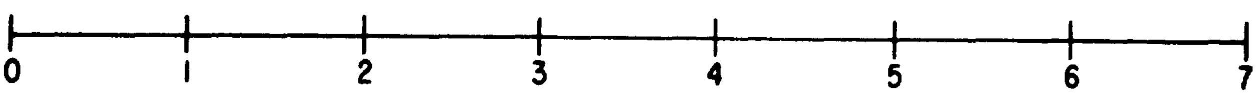

Figure 1-3.—A number line.

relationship between integers and other numbers such as fractions and decimals. It is clear that many points, other than those representing integers, exist on the number line. Examples are the points representing the numbers 1/2 (located halfway between 0 and 1) and 2.5 (located halfway between 2 and 3).

An interesting question arises, concerning the "in-between" points on the number line: How many points (numbers) exist between any two integers? To answer this question, suppose that we first locate the point halfway between 0 and 1, which corresponds to the number 1/2. Then let us locate the point halfway between 0 and 1/2, which corresponds to the number 1/4. The result of the next such halving operation would be 1/8, the next 1/16, etc. If we need more space to continue our halving operations on the number line, we can enlarge our "picture" and then continue.

It soon becomes apparent that the halving process could continue indefinitely; that is, without limit. In other words, the number of points between 0 and 1 is infinite. The same is true of any other interval on the number line. Thus, between any two integers there is an infinite set of numbers other than integers. If this seems physically impossible, considering that even the sharpest pencil point has some width, remember that we are working with ideal points, which have no physical dimensions whatsoever.

Although it is beyond the scope of this course to discuss such topics as orders of infinity, it is interesting to note that the set of integers contains many subsets which are themselves infinite. Not only are the many subsets of numbers other than integers infinite, but also such subsets as the set of all odd integers and the set of all even integers. By intuition we see that these two subsets are infinite, as follows: If we select a particular odd or even integer which we think is the largest possible, a larger one can be formed immediately by merely adding 2.

Perhaps the most practical use for the number line is in explaining the meaning of negative numbers. Negative numbers are discussed in detail in chapter 3 of this course.